M11A/1101

University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2006
Mathematics M11A
Wednesday 11 January 2006 1.30 - 3.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. Let t be a number greater than 1. Define a sequence inductively by

$$\begin{aligned} x_0 &= t \\ x_{n+1} &= \frac{x_n}{2} + \frac{t}{2x_n}, \quad \text{for } n \geqslant 0. \end{aligned}$$

Show that $x_n \geqslant \sqrt{t}$ for every n.

Show that the sequence converges, and find the limit (as a function of t).

Use x_3 (with an appropriate value of t) to find a rational approximation to $\sqrt{5}$.

2. What does it mean for a sequence to be Cauchy?

State and prove Cauchy's General Principle of Convergence.

Show that an absolutely convergent series converges.

3. Show that the following series converge.

$$\text{a)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

$$\text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$$

$$\text{c)} \sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}}$$

[You may assume standard convergence tests provided that you state them clearly.]

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4. Use the series definition of e^x to show that for $0 \leqslant x < 1$

$$1 + x \leqslant e^x \leqslant \frac{1}{1-x}.$$

For $n \geqslant 1$ define

$$s_n = \left(1 + \frac{1}{n}\right)^n \text{ and } t_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

a) Show that for each n

$$s_n \leqslant e \leqslant t_n.$$

b) What happens to the sequence $\left(\frac{t_n}{s_n}\right)$ as $n \to \infty$?

c) Deduce that $s_n \to e$ as $n \to \infty$.

5. For the purposes of this question you may assume that

$$\log x \leqslant x - 1 \quad \text{for every } x > 0. \tag{1}$$

State and prove the AM/GM inequality.

Use (1) to prove that

$$\log x \geqslant 1 - \frac{1}{x} \text{ for every } x > 0.$$

Show that if (x_i) is a sequence of positive numbers satisfying

$$\frac{1}{n}\sum_1^n x_i = 1$$

then

$$\sum x_i \log x_i \geqslant 0.$$

6. What does it mean to say that a real function defined on an interval is uniformly continuous?

Prove that a continuous function on a closed bounded interval $[a, b]$ is uniformly continuous.

Show that a uniformly continuous function on the *open* interval $(0, 1)$ is automatically bounded.

END OF PAPER